MODULE 8: RATIONAL NUMBERS, PART 5

Quotients of Decimal Fractions

Once we understand the role of the decimal point, we find that there is a remarkable similarity between how we find the quotient of two whole numbers and how we find the quotient of two decimal fractions.

Recall that a rational number is the quotient of two whole numbers. Thus $13 \div 4$ is a rational number. As a common fraction it is written as $\frac{13}{4}$ and as a mixed number it is written as $3\frac{1}{4}$. In this module we should like to see what the quotient would look like as a decimal fraction.

Perhaps the best way to start is to think in terms of money.

Example 1

The cost of a \$13 gift is to be shared equally by 4 people. How much should each person pay?

Answer: \$3.25

Since the label of our answer will be "dollars per person", we know that we must divide the cost (\$13) by the number of people (4). To this end, we may rewrite \$13 as 1,300 cents and perform the appropriate division:

We could write the answer as $\$3\frac{1}{4}$ but we want to see what this means as a decimal fraction.

$$\begin{array}{r} 325 \text{ (cents)} \\ 4\overline{)1,300} \\ -1\,2 \\ \hline 10 \\ -\ 8 \\ \hline 20 \\ -\ 20 \\ \hline 0 \end{array}$$

Since there are 100 cents per dollar, we may rewrite our answer as \$3.25

If we revisit Example 1 we find a rather nice way of obtaining the answer without first converting dollars to cents. All we have to do is write \$13 as \$13.00 and repeat exactly what we did in Example 1.

That is:

```
   $ 3.25
4)$13.00
  -12
    1 0
  -   8
     20
   - 20
      0
```

When we write 13.00 we are viewing 13 as 1,300 hundredths. So we're dividing 1,300 hundredths by 4 and the quotient is 325 hundredths--which we write as 3.25

Rule for Dividing a Decimal Fraction By a Whole Number

Write the division problem just you would have if both dividend and divisor were whole numbers.

Place the decimal point in the quotient directly above the decimal point in the dividend.

Then proceed exactly as you would have if both numbers had been whole numbers.

The divisor is still a whole number, but the dividend might not be.

If there is no decimal point in the dividend, it is assumed to come after the digit furthest to the right.

Example 2

A gift is \$29.22 and the cost is to be shared equally by 6 people, How much should each person pay?

Answer: \$4.87

We want to divide \$29.22 by 6.

Step 1: We write the quotient as

```
     .
6)$29.22
```

making sure that the decimal point in the quotient is directly above the

decimal point in the dividend.

Step 2: Now perform the division as we would had there been no decimal points:

```
    $ 4.87
  6)$29.22
    -24
      5 2
    - 4 8
       42
     - 42
        0
```

Note:

Again, what we've really done is viewed 29.22 as 2,922 hundredths. In effect, we showed that 2,922 ÷ 6 was 487. Hence 2,922 hundredths divided by 6 would be 487 hundredths, or 4.87

Check:

```
 $4.87
  X  6
$29.22
```

So we have to know how to multiply decimal fractions before we can understand ho to divide them.

Remember that we can read 4.87 as 487 hundredths because the 7 is in the hundredths-place.

Example 3

Write 3 ÷ 16 as a decimal fraction.

Answer: 0.1875

Since 3 is being divided by 16, the long division form would be $16\overline{)3}$. The "trick" now is that we can write 3 as 3.0 or 3.00 or 3.000 and so on. For example, $16\overline{)3}$ is the same thing as $16\overline{)3.000000}$. In the latter form, we can allign the decimal point in the quotient with the decimal point in the dividend and proceed as we would with whole numbers. Namely:

Use as many or as few 0's a you wish. In a few moments we'll indicate what the bes number of 0's is.

```
      0.1875
  16)3.000000
    - 1 6
      1 40
    - 1 28
        120
      - 112
         80
       - 80
          0
```

We could have continued to divide until all the 0's were used up. In this case we'd get 0.187500, which is equivalent to 0.1875

Since the remainder is now there is no need to continu

Note that in problems such as Example 3, we can annex one 0 at a time and continue until there is no remainder.

Later in this module we'll study the case of what happens if there is always a remainder, but for now we'll avoid this situation.

There is a very nice thing about decimal fractions. They never have to be reduced. In other words, while $\frac{3}{15}$ can be reduced to $\frac{1}{5}$, the decimal equivalent for $\frac{3}{15}$ is the same as the decimal equivalent for $\frac{1}{5}$.

Example 4

(a) Write $\frac{1}{5}$ as an equivalent decimal fraction.

(b) Write $\frac{3}{15}$ as an equivalent decimal fraction.

Answer: (a) 0.2 (b) 0.2

0.2 means 2 tenths or 2/10 which is equivalent to both 1/5 and 3/15.

We may proceed just as we did in our discussion of mixed numbers.

(a) $\frac{1}{5}$
```
   0.2
5)1.0
 -1 0
    0
```

Remember that the 0 in 0.2 is just to emphasize the decimal point. 0.2 and .2 mean the same thing.

(b) $\frac{3}{15}$
```
    0.2
15)3.0
  -3 0
     0
```

In particular, then, if you're going to use a calculator to find the quotient of two whole numbers, it is not necessary that you reduce to lowest terms first.

However, just as with whole numbers, you must be careful of keeping the proper place when you divide.

Example 5

Express $\frac{3}{40}$ as an equivalent decimal fraction.

Answer: 0.075

The answer is not 0.75. 0.75 means 75 hundredths, which reduces to 3/4 not 3/40.

We have: $\frac{3}{40}$
```
    0.075
40)3.0000
   2 80
    200
  - 200
      0
```

Notice that the 7 had to be placed above the <u>second</u> 0 since 30 is less than 40.

As a check notice that 0.075 means 75 thousandths or $\frac{75}{1,000} = \frac{3 \times 25}{40 \times 25} = \frac{3}{40}$.

You may have noticed that in all our examples so far we divided by numbers whose only prime factors were 2 or 5. The reason for this is closely related to what happens when we restrict ourselves to denominators that are powers of ten.

That's what happens when we use decimal fractions. The fractional places are 10ths, 100ths, 1,000ths, and so on

<u>Key Point</u>

Since the prime factorization of 10 is 2 X 5, any power of 10 can have only 2 or 5 as prime factors.

For example 100 = 10 X 10 = (2 X 5) X (2 X 5)

Let's see what happens if we try to express a common fraction as a decimal fraction if the denominator of the common fraction has a prime factor other than 2 or 5.

<u>Example 6</u>

Express $\frac{1}{3}$ as an equivalent decimal fraction.

Answer: 0.3333..... (the 3 continue endlessly)

We have:

$$\begin{array}{r} \underline{1\ \ 0.3333....} \\ 3)1.0000.... \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 10 \\ \underline{-9} \\ 1 \end{array}$$

What this means is that 1/3 cannot be expressed as an equivalent common fraction whose denominator is a powe of ten. Remember, no matte how many 0's we use, the su of the digits in 1000000... is always 1; and hence neve divisible by 3.

So the decimal equivalent of $\frac{1}{3}$ is a *non-terminating* decimal fraction. Yet the decimal fraction still has an interesting

Decimal fractions like 0.2 or 0.678947 are terminating because they eventually com to an end.

pattern. Namely, the digit 3 repeats endlessly.

> *Key Result*
>
> *If one whole number is divided by another which has only 2's and/or 5's as prime factors, then the decimal fraction form of the quotient* terminates.
>
> *If the divisor has prime factors other than just 2's or 5's, the decimal will not terminate.* However, beyond a certain place, it will repeat the same cycle of digits endlessly.

Why this must happen will be explained as we do the next few examples.

Example 7

What is the repeating cycle of digits when $3 \div 11$ is written as a decimal fraction?

Answer: The repeating cycle is 27. That is, the decimal fraction is 0.272727....

We have:

$$3 \div 11 = \frac{3}{11} = 11\overline{)3.000000}\quad 0.27$$

$$\begin{array}{r} 3.0\ \ \\ -2\,2\ \ \\ \hline 8\,0 \\ -7\,7 \\ \hline 3 \end{array} \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$$

Here's the key point. When we started the divison we were faced by a 3 followed only by 0's. Now we've arrived at the same situation. Hence we'll continue to repeat what has already happened; namely the sequence 0.272727....

The answer to Example 7 is not 0.27 or 0.2727. It is the *endless (non-terminating)* 0.272727...... To indicate that the decimal fraction is non-terminating we place a horizontal bar over the repeating cycle. That is, we write $0.\overline{27}$ as an abbreviation for 0.272727....

Sometimes we write $0.2727\overline{27}$ rather than $0.\overline{27}$ to emphasize the fact that the decimal fraction repeats the cycle 27 endlessly.

Note:

0.27 is $\frac{27}{100}$ while $0.\overline{27}$ is $\frac{3}{11}$. Since $\frac{27}{100}$ and $\frac{3}{11}$ are not equivalent, neither are 0.27 and $0.\overline{27}$ equivalent decimal fractions.

Example 8

Write $7 \div 15$ as a decimal fraction.

Answer: $0.4\overline{6}$

We have:

$$7 \div 15 = \frac{7}{15} \quad \begin{array}{r} 0.4\,6\,6 \\ 15\overline{)7.0\,0\,0\,0\,0\,0\,0} \\ -6\,0 \\ \hline 1\,0\,0 \\ -\,9\,0 \\ \hline 1\,0\,0 \\ -\,9\,0 \\ \hline 1\,0 \end{array}$$

Caution:

The 4 is not part of the repeating cycle. (see Example 9)

From this point on, the remainder 10 will continue to repeat, so that we'll get nothing but 6's in the quotient.

The reason the decimal in Example 8 didn't terminate is that the divisor, 15, has 3 as a prime factor. But why does the decimal ultimately repeat the same cycle of digits? Well, suppose you divide a number by 15. There are only 15 possible remainders; namely 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, or 14. *Hence if we have 16 zeros after the decimal point, at least one of the remainders has to repeat.* In Example 8 the remainder (10) repeated much sooner, but by the 16th place it would have had to repeat.

In terms of putting 15 book into a carton, if you have 15 books or more remaining you can fill another carton 0 is a remainder because it is possible that when th last carton is filled there are no books left.

In a similar way, if we divide by, say, 27, there are 27 possible remainders (0 through 26). Hence by the 28th place after the decimal point a remainder would have to repeat.

Of course, once 0 occurs as the remainder, 0 will alway continue to occur as the remainder--in which case th decimal terminates.

Example 9

Write $46 \div 99$ as a decimal fraction.

Answer: $0.\overline{46}$

We have:

$$46 \div 99 = \frac{46}{99} \quad \begin{array}{r} 0.4\,6\,4 \\ 99\overline{)4\,6.0\,0\,0\,0\,0\,0} \\ -\,3\,9\,6 \\ \hline 6\,4\,0 \\ -5\,9\,4 \\ \hline 4\,6\,0 \\ -\,3\,9\,6 \\ \hline \end{array}$$

In this example we start with 46 followed only by 0' and 46 is the first remaind to repeat. In Example 8, w started with 7 followed by 0's but the first repeating remainder was 10.

Since $\frac{46}{99}$ and $\frac{7}{15}$ aren't equivalent common fractions, $0.\overline{46}$ and $0.4\overline{6}$ aren't equivalent decimal fractions. So we have to be careful how we write the bar in non-terminating decimal fractions.

Non-terminating decimal fractions are often disconcerting. We aren't used to seeing numbers that don't end so the concept is troublesome. However, in the real-world we often avoid this problem by rounding off. For example, if an item is priced at 3 for \$1 or \$$\frac{1}{3}$ each, you usually pay 34¢ for an individual item. That is, the store rounds \$$0.\overline{3}$ off to \$0.34.

More precisely, the store rounds <u>up</u> to \$0.34

We round off decimals the same way as we round off whole numbers. For example, the same reasoning that tells us that 3,416 is between 3,000 and 4,000 but closer in value to 3,000 also tells us that 0.3416 is between 0.3000 and 0.4000 but closer in value to 0.3

0.3 and 0.4 are consecutive multiples of tenths. That is 0.3 is 3 tenths and 0.4 is 4 tenths.

<u>Example 10</u>

Which decimal fraction names the greater ratio, 0.113 or 0.099?

Answer: 0.113

0.113 is 113 thousandths
and
0.099 is 99 thousandths.

Since 113 thousandths is greater than 99 thousandths, 0.113 is greater than 0.099.

The 9's seem to indicate larger numbers than the 1's but place value is important 0.099 is less than 0.1 in the same sense that 1 dime has more value than 9 cents.

In terms of common fractions, $\frac{113}{1,000}$ exceeds $\frac{99}{1,000}$ by $\frac{14}{1,000}$. In terms of decimal fractions:

$$\begin{array}{r} 0.113 \\ -\ 0.099 \\ \hline 0.014 \end{array}$$

Example 10 emphasizes how we compare the size of two decimal fractions. Namely:

> *To Compare the Size of*
> *Two (or More) Decimal Fractions*
>
> *(1) Allign the decimal fractions just as you would for addition or subtraction with the decimal points in the same column.*
>
> *(2) Look for the first place in which the decimal fractions have different digits.*
>
> *(3) Then the greater digit in this place names the greater number.*

For example:

0.0010345
0.0010623
0.0010199

In this illustration it occurs in the hundred-thousandths place.

So in this case, from least to greatest the numbers are:
0.0010199
0.0010345
0.0010623

This method makes it easy for us to round off any non-terminating decimal to any degree of accuracy.

Note

*The main point is that for a terminating decimal we can read the ratio directly. For example, 0.123456 is **123,456** millionths. But there is no digit furthest to the right in a non-terminating decimal.*

See the idea? We can "pinpoint" $0.123\overline{456}$ between 0.123456 and 0.123457

Example 11

Round off $0.\overline{46}$ to the nearest tenth.

Answer: 0.5

The tenths-digit in $0.\overline{46}$ is 4. Hence $0.\overline{46}$ is between 0.4 and 0.5. The listing in the margin shows the idea in more detail.

0.400000......	
0.450000......	*0.45 is mid-*
0.464646...	*way between*
0.500000...	*0.4 and 0.5*

As in dicated in our discuss ion following Example 10, th above decimal fractions are from least to greatest. $0.\overline{46}$ is between 0.45 and 0.5 and therefore it is closer to 0.5 than to 0.4

We see that $0.\overline{46}$ is between 0.4 and 0.5 but closer to 0.5. More mechanically:

(1) Locate the tenths-place: 0.4646....

(2) Look at the digit to its right: 0.4646...

(3) Since it's 6, change the 4 to 5 and replace the remaining digits by 0's: That is, 0.5000... or simply 0.5.

The exact value of $0.\overline{46}$ is $\frac{46}{99}$. So the difference between $0.\overline{46}$ and 0.5 is given by:

$$0.5 - 0.\overline{46} =$$
$$\frac{5}{10} - \frac{46}{99} =$$
$$\frac{1}{2} - \frac{46}{99} =$$
$$\frac{99}{198} - \frac{92}{198} =$$
$$\frac{99 - 92}{198} =$$
$$\frac{7}{198}$$

Recall that 0.5 is 5 tenths or 1 half.

So the error caused by the rounding off is 7 parts in 198

If we round off to the nearest hundredth we expect an even smaller rounding off error.

Example 12

Round off $0.\overline{46}$ to the nearest hundredth.

Answer: 0.46

The fact that $0.\overline{46} = 0.46....$ tells us that $0.\overline{46}$ is between 0.46 and 0.47. 0.465 is midway between 0.46 and 0.47. So we have:

$$\begin{aligned} 0.46 &= 0.46000000..... \\ 0.\overline{46} &= 0.46464646..... \\ 0.465 &= 0.46500000..... \\ 0.47 &= 0.47000000..... \end{aligned}$$

So we see that $0.\overline{46}$ is between 0.46 and 0.465, hence closer to 0.46 than to 0.47.

Alternatively, locate the hundredths-place in $0.\overline{46}$ and look at the digit to its right.

0.464646....

This digit (4) is less than 5 so replace it and each digit to its right by 0's.

0.4600000....

or 0.46

To see how close our new approximation is, notice that 0.46 is $\frac{46}{100}$ or $\frac{23}{50}$. Hence:

$$\begin{aligned} 0.\overline{46} - 0.46 &= \frac{46}{99} - \frac{23}{50} \\ &= \frac{(46X50) - (23X90)}{99X50} \\ &= \frac{2,300 - 2,277}{4,950} \\ &= \frac{23}{4,950} \end{aligned}$$

So now our error is 23 parts in 4,950 which is much less than our previous error of 7 parts in 198. In fact at a rate of 7 parts per 198 we'd have:

$$\begin{array}{rr} & 7 \text{ per } 198 \\ + & 7 \text{ per } 198 \\ + & 7 \text{ per } 198 \\ + & \underline{7 \text{ per } 198} \\ & 28 \text{ per } 792 \end{array}$$

In other words, we multipl[y] both 7 and 198 by 4.

which is much greater than only 23 per 4,950.

We can continue to round off to greater and greater accuracy until any error is less than what we're willing to accept.

Sometimes an error of 7 per 198 is considered goo[d]. Other times an error of 23 per 4,950 is considered bad. How much accuracy we require is dictated by the situation.

Now that we know how to express the quotient of two whole numbers as a decimal fraction, we turn our attention to the question of expressing the quotient of any two *decimal fractions* as a decimal fraction. For example, we might be interested in expressing 0.02 ÷ 0.004 as a decimal fraction. One way to do this would be to express each decimal as an equivalent common fraction and do the resulting problem.

Example 13

Write 0.02 ÷ 0.004 as a decimal fraction.

Answer: 5 or 5. or 5.0 etc

$$0.02 = \frac{2}{100} = \frac{1}{50}$$

$$0.004 = \frac{4}{1,000} = \frac{1}{250}$$

Hence:

$$\begin{aligned} 0.02 \div 0.004 &= \frac{1}{50} \div \frac{1}{250} \\ &= \frac{1}{50} \times \frac{250}{1} \\ &= \frac{250}{50} \end{aligned}$$

We've translated an unfamiliar problem to a more familiar one.

$$= \frac{5 \text{ X } 50}{1 \text{ X } 50}$$

$$= \frac{5}{1}$$

$$= 5$$

As a decimal fraction we might write 5 as 5. or 5.0 or 5.00 but this distinction is not crucial.

Note
It may seem strange that with numbers as small as 0.02 and 0.004 that the quotient is as great as 5. But recall that the quotient gives us the size of one number relative to the other. In other words, 0.02 may seem like a small number but compared with 0.004 it is 5 times as great.

That is $0.004 \text{ X } 5 = 0.020$
$= 0.02$

There is another way of doing Example 13 which doesn't require our translating into the language of common fractions first.

Key Point
In any ratio we can multiply both numbers by the same non-zero number without changing the ratio. In other words if we multiply the dividend and the divisor by the same amount, the quotient remains the same.

In the language of common fractions, this says we can multiply numerator and denominator by the same non-zero number and get an equivalent common fraction.

If we multiply 0.02 by 1,000 we get 20 (that is, we move the decimal point 3 places to the right) and if we multiply 0.004 by 1,000 we get 40. Hence:

$$0.02 \div 0.004 = (0.02 \text{ X } 1{,}000) \div (0.004 \text{ X } 1{,}000)$$
$$= 20 \div 4$$

That is, 0.02 is as many times greater than 0.004 as 20 is greater than 4.

In common-fraction form:

$$0.02 \div 0.004 = \frac{0.02}{0.004}$$
$$= \frac{0.02 \text{ X } 1{,}000}{0.004 \text{ X } 1{,}000} = \frac{20}{4}$$

Since multiplying or dividing by a power of ten simply involves moving the decimal point a certain number of places, we have the following rule:

To Convert a Quotient of Decimal Fractions into a Quotient of Whole Numbers:

In dividing one decimal fraction by another, we may move the decimal point in each the same number of places (to the right) without changing the ratio.

So we move the decimal point enough places to convert each decimal fraction into a whole number.

Example 14

Which is the greater ratio:

$0.023 \div 0.46$ or $23 \div 460$?

Answer: They're equal

Method 1 (Converting to Common Fractions)

$0.023 = \frac{23}{1,000}$ and 0.46 is $\frac{46}{100}$

Hence:

$$
\begin{aligned}
0.023 \div 0.46 &= \frac{23}{1,000} \div \frac{46}{100} \\
&= \frac{23}{1,000} \times \frac{100}{46} \\
&= \frac{2,3\cancel{00}}{46,0\cancel{00}} \\
&= \frac{23}{460} \\
&= 23 \div 460
\end{aligned}
$$

This could be reduced more but we want to compare it with 23 ÷ 460

Method 2 (Moving the Decimal Point)

To convert 0.023 to a whole number we must move the decimal point at least 3 places to the right.

If we move it 2 places we get 2.3; 3 places gives us 23; and 4 places gives us 230.

To convert 0.46 to a whole number, we must move the decimal point at least 2 places to the right.

We must move the decimal point the same number of places in each number. So to make sure that both numbers are whole numbers, we move each decimal point 3 places to the right to get:

$$\frac{0.023}{0.46} = \frac{0.023 \text{ X } 1{,}000}{0.46 \text{ X } 1{,}000}$$
$$= \frac{23}{460}$$

If we moved it 4 places in each we'd get 230/4,600 which reduces to the same ratio

Note:

If we wanted to write 0.023 ÷ 0.46 in decimal form, we could reduce 23/460 to 1/20 and get:

```
 1   0.05
 20)1.00
   - 0
     1 00
   - 1 00
        0
```

We don't have to reduce to lowest terms. Namely:

```
 23     0.0 5
 460) 2 3.0 0
          0
        2 3 0 0
      - 2 3 0 0
```

Example 15

Write 28.42 ÷ 1.4 as a decimal fraction.

Answer: 20.3

Move the decimal point 2 places to the right in each number to obtain:

28.42 ÷ 1.4 =

2,842 ÷ 140

and proceed as we did earlier in this module:

```
          2 0. 3
 140)2 8 4 2. 0 0 0
    -2 8 0
         4 2
       - 0
         4 2  0
        -4 2  0
              0
```

"Plausiblity Check"

28.42 ÷ 1 = 28.42
28.42 ÷ 2 = 14.21
So 28.42 ÷ 1.4 should be between 14.21 and 28.42; and 20.3 is in this range.

Actually the crucial part is for the divisor to be a whole number. So we could have written:

```
      2 0.3
 14)2 8 4.2
    2 8
        4
      - 0
        4 2
      - 4 2
          0
```

In other words, we only had to move the decimal point 1 place to the right in both 28.42 and 1.4

Don't lose track of the fact that division is still a special form of multiplication.

Example 16

What decimal fraction must we multiply by 1.4 to get 28.42 as the product?

Answer: 20.3

In terms of fill-in-the-blank we have:

$$\_\_\_\_\_ \times 1.4 = 28.42$$

and this is the same as _____ = 28.42 ÷ 1.4;

which is exactly the same problem we solved in Example 15.

So it sounds like a multiplication problem but it's really a division problem.

Before we conclude this module it is worth noting that decimal fractions are frequently used in dealing with percents.

Example 17

Write 0.3% as a common fraction in lowest terms.

Answer: $\frac{3}{1,000}$

0.3 is the same as $\frac{3}{10}$ and % still means to divide by 100. Hence:

$$\begin{aligned} 0.3\% &= \frac{3}{10} \div 100 \\ &= \frac{3}{10} \times \frac{1}{100} \\ &= \frac{3}{1,000} \end{aligned}$$

From a more intuitive point of view, 0.3% means
0.3 per 100
and this is the same rate a
3 per 1,000

But using decimal fractions there is an easier way to divide by 100. All we have to do is move the decimal point 2 places to the left. Hence 0.3% becomes 0.003, which is the same as $\frac{3}{1,000}$

So now we may find percents even when they're given in decimal form.

Example 18

How much is 27.2% of 18,000?

Answer: 4,896

27.2 is between 25 and 30. So 27.2% of 18,000 is between 25% of 18,000 and 30% of 18,000. That is, our answer is between 4,500 and 5,400

Method 1

$27.2\% = 27.2 \div 100$

$= 0.272$

Hence:

27.2% of 18,000 =

$0.272 \times 18{,}000 =$

4,896

Method 2 (Converting to Common Fractions)

$$27.2 = 27\frac{2}{10} = 27\frac{1}{5} = \frac{(27 \times 5) + 1}{5} = \frac{136}{5}$$

Therefore 27.2% is equal to $\frac{136}{500}$

$$\text{and } \frac{136}{500} \text{ of } 18{,}000 = \frac{136 \times 18{,}\cancel{000}}{5\cancel{00}} = \frac{136 \times 36}{1} = 4{,}896$$

Remember that to convert a percent to a fraction, we divide by 100. In terms of common fractions, we divide by 100 by annexing two 0's to the denominator.

In Example 18, Method 2 seems much more complicated than does Method 1. *However in the case where we'd have a non-terminating decimal fraction, Method 2 is extremely important if we desire the exact answer.*

Example 19

How much is $27\frac{1}{3}\%$ of 18,000?

Answer: 4,920

$$27\frac{1}{3} = \frac{(27 \times 3) + 1}{3} = \frac{82}{3}$$

Therefore $27\frac{1}{3}\% = \frac{82}{300}$

So:

$$27\frac{1}{3}\% \text{ of } 18{,}000 =$$

$$\frac{82}{300} \text{ X } 18{,}000 =$$

$$82 \text{ X } \frac{18{,}000}{300} =$$

$$82 \text{ X } 60 =$$

$$4{,}920$$

27 1/3% means 27 1/3 per 1(
and at this rate, we're taking 82 per 300.

But if you wrote $27\frac{1}{3}\%$ as 27.33% you'd get that $27\frac{1}{3}\%$ of 18,000 =

$$0.2733 \text{ X } 18{,}000 =$$

$$4919.4$$

Recall that $\frac{1}{3}$ is $0.\overline{3}$ not 0.3 or 0.33 or 0.333 etc.

While this is close to the right answer, it is not exactly correct. The reason is that we have rounded $27\frac{1}{3}\%$ off to 27.33%.

We're only off by 0.6 in 18,000 but if we were deal ing with millions of items this could be a very signi ficant error.

For example an error of 1 inch in a mile could resul in an error of more than 1,450 miles by the time we got to the sun.

In any event this completes our treatment of rational numbers. In the next few modules we shall apply what we've learned to some practical situations, We can't, of course, cover all possible applications, but by the time we finish you should have a good idea of how mathematics is used in almost all walks of our everyday living.

Appendix 1: More on Non-Terminating, Repeating Decimals

In this module we saw that any quotient of whole numbers led to a decimal fraction that either terminated or repeated the same cycle of digits endlessly. A natural question is whether the reverse is also true.

Main Question:

> *If a decimal fraction either terminates or repeats the same cycle of digits endlessly, does it represent the quotient of two whole numbers?*

For example, is $0.\overline{135}$ the quotient of two whole numbers? As we shall soon see, the answer is "Yes!"

The answer to this question is *yes*. The easiest case to analyze is when the decimal terminates. In fact, we've already treated this case in the previous module. Namely:

> *To Write a Terminating Decimal Fraction as The Quotient of Two Whole Numbers.*
>
> *Look at the terminating decimal fraction and write the number you get if you omit the decimal point. This becomes the numerator of the equivalent common fraction.*
>
> *Then write a 1 followed by as many 0's as there are digits to the right of the decimal point. This becomes the denominator.*
>
> *The numerator divided by the denominator is then one desired pair of whole numbers.*

The fact that the decimal terminates guarantees that there is a final digit.

Example 20

Write 0.9325 as the quotient of two whole numbers.

Answer: One such quotient is $9,325 \div 10,000$

The numerator of the equivalent common fraction is 9,325 and the denominator is 10,000. Hence one such common fraction is

$$\frac{9,325}{10,000}$$

That is, if we omit the decimal point we get 9325 and there are four 0's in the denominator because there are four digits to the right of the decimal point.

and this can be reduced to:

$$\frac{25 \text{ X } 373}{25 \text{ X } 400}$$

or $\frac{373}{400}$.

Hence 373 ÷ 400 is one such quotient and 9,325 ÷ 10,000 is another.

Remember that there are many common fractions that are equivalent to another. To make sure that we all use the same form, we often accept the custom of using the quotient that is in lowest terms.

The harder case is when the decimal fraction is non-terminating. In this case, we keep track of the repeating cycle. For example, look at $0.\overline{135}$. The repeating cycle is 135. That is, the cycle 135 repeats endlessly after the decimal point. *So if we move the decimal point 3 places to the right, the same cycle, 135, will again repeat. Moving the decimal point 3 places to the right multiplies the decimal fraction by 1,000. So if we let D denote the decimal, we have:*

That is: 0.135135135135...
135.135135135..

$$\begin{array}{rl} 1{,}000 \text{ X } D = & 135.135135..... \\ - \quad D = & 0.135135..... \\ \hline 999 \text{ X } D = & 135.000000..... \end{array}$$

or

$$999 \text{ X } D = 135$$

and this is the same as saying that

$$D = 135 \div 999$$
$$= \frac{135}{999}$$
$$= \frac{\cancel{3} \text{ X } \cancel{3} \text{ X } \cancel{3} \text{ X } 5}{\cancel{3} \text{ X } \cancel{3} \text{ X } \cancel{3} \text{ X } 37}$$
$$= \frac{5}{37}$$

1,000 X D is the same as the sum of 1,000 D's and i
we subtract (one) D from 1,000 D's we have 999 D's left and this means 999 X

The trick was that by movi the decimal point 3 places to the right, the two frac tional parts were the same so they cancelled out when we subtracted.

As a check we see that:

```
      5 0.1 3 5 1
   37)5.0 0 0 0
     -3 7
      1 3 0
    - 1 1 1
        1 9 0
      - 1 8 5
            5 0
          - 3 7
```

We're now back to the same sequence (a 5 followed only by 0's) that we had at the beginning. Hence 135 will be the repeating sequence.

In other words, if we know that the decimal is $0.\overline{135}$, we can show that it represents the quotient:

$$5 \div 37,$$

a fact that may not seem that obvious just by looking at $0.\overline{135}$.

Example 21

Write $0.\overline{23}$ as the quotient of two whole numbers.

Answer: $0.\overline{23} = 23 \div 99$

Let D stand for $0.\overline{23}$. That is:

D = 0.23232323.........

Note that the answer is NOT $23 \div 100$. $23 \div 100$ is 0.23, which is not the same as 0.232323....

The same cycle repeats if we move the decimal point 2 places to the right; that is, if we multiply by 100. Hence:

```
  100 D's = 23.2323.....
  -   D   =  0.2323.....
  ----------------------
   99 D's = 23.
```

or 99 X D = 23; hence D = 23 ÷ 99

By moving the decimal point 2 places to the right, the "endless parts" cancel when we subtract.

```
 23    0. 2 3 2 3 .....
 99)2 3. 0 0 0 0 . . .
   -1 9  8
      3  2 0
   -  2  9 7
         2 3 0
       - 1 9 8
           3 2 0
         - 2 9 7 and so on...
```

Sometimes it takes a "few" digits before the decimal fraction begins to repeat. In such a case, we still look for the repeating cycle. Let's look at one such illustration. Suppose D = $0.1\overline{35}$. In this case the 1 is not part of the repeating cycle because the bar is only over the cycle 35.

That is, $D = 0.135353535......$

Since 35 is the repeating cycle, the fewest number of places we can move the decimal point to emphasize this cycle is 1 to the right, which is the same as multiplying by 10. That is:

$$10 \text{ D's} = 1.353535..... \qquad (1)$$

If we had moved the decimal point in D 3 places to the right (that is, if we multiplied by 1,000) the 35 cycle would repeat again. That is:

$$1{,}000 \text{ D's} = 135.353535... \qquad (2)$$

Now if we subtract (1) from (2) we get:

$$\begin{array}{rrl} & 1{,}000 \text{ D's} = & 135.353535..... \\ - & 10 \text{ D's} = & 1.353535..... \\ \hline & 990 \text{ D's} = & 134. \end{array}$$

Our strategy was to move the decimal points in such a way that the fractional parts would cancel when we subtracted.

So $D = \frac{134}{990}$ or $\frac{67}{495}$

Check:

```
           0.1 3 5 3 5 . . .
     495)6 7.0 0 0 0 0 . . .
        -4 9 5
         1 7 5 0
        -1 4 8 5
           2 6 5 0
         - 2 4 7 5
             1 7 5 0
           - 1 4 8 5
               2 6 5 0
             - 2 4 7 5
                 1 7 5
```

By now you should begin to see the repeating cycle 35 emerge.

We'll supply more practice in the Self-Test. But for now we should sense that every terminating or non-terminating but repeating decimal fraction expresses the quotient of two whole numbers. In particular if a decimal fraction doesn't terminate nor repeat the same cycle endlessly, it cannot

represent the quotient of two whole numbers. Such a decimal fraction is called an *irrational number*.

That is, the quotient of two whole numbers is called a rational number. So if a decimal cannot be the quotient of two whole numbers we call it an irrational number.

It is easy to write irrational numbers. For example, suppose we started with 0.23 and each time we repeated the cycle we added another 3. We'd get:

0.23233233323333233333.....

Since we add another 3 to each cycle it is impossible for the same cycle to repeat endlessly.

That is, each cycle has one more 3 than the previous cycle. Hence it is a different cycle.

Irrational numbers play an important part in the real world of science and technology. But when we make measurements we can always approximate an irrational number by "trapping" it between two rational numbers.

For example, if we let I stand for 0.23233233323333.... notice that I is between 0.2 and 0.3. If we want a better approximation it is between 0.23 and 0.24. A still better approximation is that I is between 0.232 and 0.233 and so on.

Do you see how "and so on" works here? We read the first "few" places of I to get, say, 0.23233, and then we increase the last digit we wrote by 1 to get 0.23234. Hence I is between 0.23233 and 0.23234; which are rational numbers because they terminate. In most cases, such an approximation is more than adequate.

"Squeezing" a number between two others, involves what we call *inequalities*. Inequalities will be discussed in an introductory way in the next appendix.

Because irrational numbers can always be approximated adequately be rational numbers, we prefer not to pursue irrational numbers further in this course.

Appendix 2: Inequalities and Rounding Off

Many times in life we are more interested in inequalities than in equalites. For example, a sign on a bridge might say that the bridge is unsafe for weights of more than ten tons. It will never say that the bridge in unsafe only for a weight of ten tons. We might talk about not paying more than $100 for an item. We rarely say that we want to pay exactly $100 for an item.

Just as we have a special symbol (=) to denote equality we also have special symbols to denote inequalities.

The equal sign consista of two parallel lines. This means that the space between the two lines is the same at both ends. Symbolically this is a nice way to denote equalities. To indicate that one number was less than another we might "slant" the lines so that the smaller opening was beside the smaller number. That is:

See what happens? The smaller number is next to the smaller space. The greater number is next to the greater space.

To make sure that we don't confuse the inequality symbol with a "sloppy" equal sign, we close the lines completely at the smaller end. That is, the smaller space is no space at all. Once the ends of the lines are pinched together, the symbol looks like an "arrow head"; for example < or >. Either way, the

lesser number goes next to the closed part. *In terms of the arrowhead, the arrowhead always points to the lesser number.*

This is because the closed end is the lesser space.

> ***New Symbolism***
>
> *We read $c < d$ as "c is less than d". If we want to indicate that d is more (greater) than c, we'd write $d > c$.*

See? The arrow points to the smaller number.

The arrow still points to the smaller number.

Example 22

Which is correct, $3 > 4$ or $4 > 3$?

Answer: $4 > 3$

The arrow has to point to the lesser number. Since 3 is less than 4, the arrow must point to 3. Hence $4 > 3$ is correct.

Alternatively, the lesser number must be next to the smaller space (that is, the closed end).

Example 23

Do $5 > 2$ and $2 < 5$ mean the same thing?

Answer: Yes

In both cases the arrow points to the 2. Hence either way we're saying that 5 is greater than 2. In the first case, the wording would be "5 is greater than 2" The second wording would be "2 is less than 5".

So as long as the arrow points to the smaller number, we can write the numbers in either order.

The arithmetic of inequalities has many applications but as far as our course is concerned the most immediate use is for making estimates and rounding off.

For example, when we wanted to round off 534 to the nearest 100, we located 534 between two consecutive multiples of 100; namely 500 and 600. In the

language of inequalities, we could write:

$$500 < 534 < 600$$

The first arrow points to the 500, so we read it as "500 is less than 534". The second arrow points to the 534, so we read it as "600 is greater than 534". If we want to read it in the order that emphasizes 534, we say that 534 is less than 600 but greater than 500.

Then since 534 is closer to 500 than to 600, we round 534 off to 500.

Example 24

Use the inequality symbols to indicate that 534 is between the 53rd multiple of 10 and the 54th multiple of 10.

Answer: $530 < 534 < 540$

The 53rd multiple of 10 is 530 and the 54th multiple of 10 is 540. Since 534 is less than 540 but greater than 530, we can write:

$$530 < 534 < 540$$

Note that there are other ways of writing the answer. For example we could say that:

$534 > 530$ and $540 > 534$

but $530 < 534 < 540$ seems easiest to read correctly.

In estimating 3.14 X 2.7 (Example 21, Module 7) we could have used inequality symbols. Namely:

$$\text{(X)}\quad \begin{array}{ccccc} 3 & < & 3.14 & < & 4 \\ \underline{2} & < & 2.7 & < & \underline{3} \\ 6 & & & & 12 \end{array}$$

and this tells us that 3.14 X 2.7 is between 6 and 12.

Example 25

By locating 5.23 and 4.8 between consecutive whole numbers, use the inequality symbols to find the range 5.23 X 4.8 is in.

Answer: $20 < 5.23 \times 4.8 < 30$

That is, the product is between 20 and 30.

5.23 is between 5 and 6 while 4.8 is between 4 and 5. In terms of inequalities we have:

$$\begin{array}{rrcrcr} & 5 & < & 5.23 & < & 6 \\ (\mathrm{X}) & 4 & < & 4.8 & < & 5 \\ \hline & 20 & < & ? & < & 30 \end{array}$$

Check:

$$\begin{array}{r} 5.23 \\ \times\ 4.8 \\ \hline 4184 \\ 2092 \\ \hline 25.104 \end{array}$$

To the nearest whole number 5.23 rounds off to 5 and 4.8 rounds off to 5. Hence the product rounds off to 25. Putting everything together, we can say that the product is between 20 and 30, but close to 25. This strongly agrees with the exact answer, 25.104

So the nice thing about our approximation is that we know before we start to look for the exact answer, that it's between 20 and 30. At the very least this prevents us from an error such as writing 2.5104. Namely 2.5104 is not between 20 and 30.

Additional drill is left for the Self-Test and for later modules. For now our treatment of rational numbers is sufficient to allow us to pursue more practical applications of arithmetic. This will be the goal of the next few modules.

From this point on, you will be encouraged to do as much arithmetic as possible with a calculator. For this reason, make sure you do the Cumulative Self-Test at the end of Module 8 in the Study Guide. It is the last time we will check whether you know the computational skills of arithmetic.

From this point on, we'll be more interested in whether you can <u>use</u> arithmetic than in whether you can do it!